4D neutral signature VSI and CSI spaces

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Abstract

In this paper we present a number of four-dimensional neutral signature exact solutions for which all of the polynomial scalar curvature invariants vanish (VSI spaces) or are all constant (CSI spaces), which are of relevence in current theoretical physics.

1 Introduction

In [1] we considered pseudo-Riemannian spaces of arbitrary signature for which all of their polynomial curvature invariants vanish (VSI spaces). We discussed an algebraic classification of pseudo-Riemannian spaces in terms of the boost weight decomposition and defined the $\mathbf{S}_i$- and $\mathbf{N}$-properties, and showed that if the curvature tensors of the space possess the $\mathbf{N}$-property then it is a VSI space. We also showed that the VSI spaces constructed possess a geodesic, expansion-free, shear-free, and twist-free null-congruence and hence are pseudo-Riemannian Kundt metrics of arbitrary signature.

We will use this fact to construct a set of four-dimensional (4D) neutral signature metrics, in which all of the scalar curvature invariants vanish, generalizing the 4D degenerate Kundt metrics [3] and the VSI metrics [2] in the Lorentzian case. We will also consider 4D neutral Einstein metrics which for which all of the curvature invariants are constant [5, 6] (CSI spaces), and present two simple examples for illustration.

All of these 4D neutral signature solutions are of interest in the twistor approach to string theory [7] and for spaces admitting parallel spinors [8] (where some VSI spaces are already known).

2 Pseudo-Riemannian Kundt metrics

Let us first define the pseudo-Riemannian Kundt metrics in arbitrary signature and arbitrary dimension [1]. Consider a null-vector $\boldsymbol{\ell}$ with an associated null-vector $\mathbf{n}$ so that $\ell_\mu n^\mu = 1$. We then form the null-frame $\{\boldsymbol{\ell}, \mathbf{n}, \boldsymbol{\omega}^i\}$ and define $L^{ij} = \ell^{\mu;\nu}\omega_\mu{}^i\omega_\nu{}^j$, where $\omega_\mu{}^i$ is the vielbein corresponding to $\boldsymbol{\omega}^i$ and, as usual, a semi-colon denotes covariant differentiation. We call $\boldsymbol{\ell}$ geodesic if and only if (iff) $\ell^\mu\ell_{\nu;\mu} = 0$, twist-free iff $L_{[ij]} = 0$, expansion-free iff $L^i{}_i = 0$, and shear-free if $L_{(ij)} = 0$.

Definition 2.1. A pseudo-Riemannian space is a Kundt metric if it possesses a non-zero null vector $\boldsymbol{\ell}$ which is geodesic, expansion-free, twist-free and shear-free.

We will consequently consider such metrics in null coordinates:

$$\mathrm{d}s^2 = 2\mathrm{d}u\left[\mathrm{d}v + H(v,u,x^C)\mathrm{d}u + W_A(v,u,x^C)\mathrm{d}x^A\right] + g_{AB}(u,x^C)\mathrm{d}x^A\mathrm{d}x^B, \quad (1)$$

where the range of A is over $n-2$ coordinates The metric (1) possesses a null vector field $\boldsymbol{\ell}$ obeying

$$\ell_{\mu;\nu} = L_{11}\ell_\mu\ell_\nu + L_{1i}\ell_{(\mu}\omega^i{}_{\nu)},$$

and consequently, ℓ_μ is geodesic, non-expanding, shear-free and non-twisting. In particular,

$$\ell^\mu\ell_{\nu;\mu} = \ell^\mu{}_{;\mu} = \ell^{\nu;\mu}\ell_{(\nu;\mu)} = \ell^{\mu;\nu}\ell_{[\mu;\nu]} = 0. \quad (2)$$

2.1 Walker metrics

A special case of the Kundt metrics is the Walker metrics. If $\boldsymbol{\ell}^1$ and $\boldsymbol{\ell}^2$ are two mutually orthogonal null vectors spanning a 2-dimensional invariant null plane, then they satisfy the recurrence condition

$$[\ell^1_{[a}\ell^2_{b]}]_{;c} = [\ell^1_{[a}\ell^2_{b]}]k_c, \quad (3)$$

for some recurrence vector k_c (and the two null vectors are automatically surface forming). We can interpret this as requiring that the bivector, $\boldsymbol{\ell}^1 \wedge \boldsymbol{\ell}^2$ of the invariant 2-plane is recurrent. A pseudo-Riemannian space admitting a 2-dimensional (or 1-dimensional) invariant null plane is called a Walker space [4].

Furthermore, Walker [4] showed that the metric can be written in the canonical form:

$$\mathrm{d}s^2 = \mathrm{d}u^I(2\delta_{IJ}\mathrm{d}v^J + B_{IJ}\mathrm{d}u^J + H_{Ii}\mathrm{d}x^i) + A_{ij}\mathrm{d}x^i\mathrm{d}x^j, \quad (4)$$

where B_{IJ} is a symmetric matrix which may depend on all of the coordinates, while H_{Ii} and A_{ij} do not depend on the coordinates v^I. This Walker metric admits a null-vector $\boldsymbol{\ell}^1$ (in fact two, $\boldsymbol{\ell}^1$ and $\boldsymbol{\ell}^2$) which is geodesic, expansion-free, shear-free and twist-free, and is hence a pseudo-Riemannian Kundt metric.

3 4D Neutral-signature case

A space is VSI if it satisfies the $\mathbf{S}_1$-property and the $\mathbf{N}$-property [1], and assuming a Kundt metric (1), this implies that the Riemann components $R_{010A} = -(1/2)W_{A,vv} = 0$ and $R_{0101} = \sigma = 0$, and the matrices

$$\left[\mathsf{a}^A{}_B\right], \quad \left[\mathsf{s}^A{}_B\right], \quad \left[\widetilde{R}^{AB}{}_{CD}\right],$$

in eqns. (20)-(22) for the components of the Riemann tensor in [1] are nilpotent (i.e., they have only zero-eigenvalues). In particular, assuming $W_{A,vv} = 0$, these quantities are related to the Riemann components as follows: $R_{01AB} = \mathsf{a}_{AB}$, $R_{0(A|1|B)} = (1/2)\mathsf{s}_{AB}$ and $R^{AB}{}_{CD} = \widetilde{R}^{AB}{}_{CD}$.

Let us consider the 4D neutral $(2,2)$-signature case. Here we can write

$$\mathrm{d}s^2 = 2\left(\boldsymbol{\ell}^1 \otimes \boldsymbol{n}^1 + \boldsymbol{\ell}^2 \otimes \boldsymbol{n}^2\right). \tag{5}$$

There is only one independent component of the "transverse Riemann tensor" $\widetilde{R}^{\hat{A}\hat{B}}{}_{\hat{C}\hat{D}}$ and so, requiring the $\mathbf{N}$-property, implies that this must be flat space. Therefore, we can write:

$$2\boldsymbol{\ell}^2 \otimes \boldsymbol{n}^2 = 2\mathrm{d}u^2 \otimes \mathrm{d}v^2 = -\mathrm{d}T \otimes \mathrm{d}T + \mathrm{d}X \otimes \mathrm{d}X.$$

We can then find two classes of pseudo-Riemannian Kundt VSI metrics, which can be written:

$$\mathrm{d}s^2 = 2\mathrm{d}u^1 \otimes \left(\mathrm{d}v^1 + H\mathrm{d}u^1 + W_{\mu_1}\mathrm{d}x^{\mu_1}\right) + 2\mathrm{d}u^2 \otimes \mathrm{d}v^2, \tag{6}$$

where the 1-form W_{μ_1,v^1} can be null, or timelike/spacelike. The two classes are:

Null case:

$$\begin{aligned} W_{\mu_1}\mathrm{d}x^{\mu_1} &= v^1 W^{(1)}_{u^2}(u^1,u^2)\mathrm{d}u^2 + W^{(0)}_{u^2}(u^1,u^2,v^2)\mathrm{d}u^2 + W^{(0)}_{v^2}(u^1,u^2,v^2)\mathrm{d}v^2, \\ H &= v^1 H^{(1)}(u^1,u^2,v^2) + H^{(0)}(u^1,u^2,v^2), \end{aligned} \tag{7}$$

Spacelike/timelike case:

$$\begin{aligned} W_{\mu_1}\mathrm{d}x^{\mu_1} &= v^1 W^{(1)}\mathrm{d}X + W^{(0)}_T(u^1,T,X)\mathrm{d}T + W^{(0)}_X(u^1,T,X)\mathrm{d}X, \\ H &= \frac{(v^1)^2}{8}\left(W^{(1)}\right)^2 + v^1 H^{(1)}(u^1,T,X) + H^{(0)}(u^1,T,X), \end{aligned} \tag{8}$$

and

$$W^{(1)} = -\frac{2\epsilon}{X}, \text{ where } \epsilon = 0, 1. \tag{9}$$

We note that these possess an invariant null-line if $W^{(1)} = 0$, and a 2-dimensional invariant null-plane (Walker space) if $W^{(0)}_{v^2} = 0$ for the null case (in order for the spacelike/timelike case to possess an invariant null 2-plane, it needs to be a special case of the null case.)

3.1 Coordinate transformations

The form of the metric (6) is invariant under the following transformations in the null case:

$$\left(u'^1, v'^1, u'^2, v'^2\right) = \left(u^1, v^1 + h(u^1, u^2, v^2), u^2, v^2\right) \tag{10}$$

$$\begin{aligned}
H'^{(1)} &= H^{(1)} \\
H'^{(0)} &= H^{(0)} - hH^{(1)} - h_{,u^1} \\
W^{(1)}_{u'^2} &= W^{(1)}_{u^2} \\
W^{(0)}_{u'^2} &= W^{(0)}_{u^2} - hW^{(1)}_{u^2} - h_{,u^2} \\
W^{(0)}_{v'^2} &= W^{(0)}_{v^2} - h_{,v^2}
\end{aligned}$$

$$\left(u'^1, v'^1, u'^2, v'^2\right) = \left(g(u^1), v^1/g_{,u^1}, u^2, v^2\right) \tag{11}$$

$$\begin{aligned}
H'^{(1)} &= (H^{(1)} + g_{,,u^1})/(g_{,u^1})^2 \\
H'^{(0)} &= H^{(0)}/(g_{,u^1})^2 \\
W^{(1)}_{u'^2} &= W^{(1)}_{u^2} \\
W^{(0)}_{u'^2} &= W^{(0)}_{u^2}/g_{,u^1} \\
W^{(0)}_{v'^2} &= W^{(0)}_{v^2}/g_{,u^1}
\end{aligned}$$

Similarily, for the spacelike/timelike case:

$$\left(u'^1, v'^1, X', T'\right) = \left(u^1, v^1 + h(u^1, X, T), u^2, v^2\right) \tag{12}$$

$$\begin{aligned}
H'^{(1)} &= H^{(1)}\left(1 - h\left(W^{(1)}\right)^2/4\right) \\
H'^{(0)} &= H^{(0)} - hH^{(1)} - h_{,u^1} + \left(hW^{(1)}\right)^2/8 \\
W^{(0)}_{X'} &= W^{(0)}_{X} - hW^{(1)} - h_{,X} \\
W^{(0)}_{T'} &= W^{(0)}_{T} - h_{,T} \\
W'^{(1)} &= W^{(1)}
\end{aligned}$$

$$\left(u'^1, v'^1, X', T'\right) = \left(g(u^1), v^1/g_{,u^1}, X, T\right) \tag{13}$$

$$\begin{aligned}
H'^{(1)} &= (g_{,u^1}H^{(1)} + g_{,,u^1})/(g_{,u^1})^2 \\
H'^{(0)} &= H^{(0)}/(g_{,u^1})^2 \\
W^{(0)}_{X'} &= W^{(0)}_{X}/g_{,u^1} \\
W^{(0)}_{T'} &= W^{(0)}_{T}/g_{,u^1} \\
W'^{(1)} &= W^{(1)}/g_{,u^1}
\end{aligned}$$

4 Exact VSI solutions

Let us present the exact VSI solutions in the null and non-null cases.

4.1 Null case

Consider first the null case using the form of the metric eq. (7). Computing the Ricci tensor, the conditions for the VSI space to be a vacuum, $R_{ab} = 0$, yields the following system of partial differential equations:

$$
\begin{aligned}
0 = v^1 &\left(W^{(1)}_{u^2} \frac{\partial}{\partial v^2} H^{(1)} - 2 \frac{\partial^2}{\partial v^2 \partial u^2} H^{(1)} \right) \\
&- 2 \frac{\partial^2}{\partial v^2 \partial u^2} H^{(0)} - W^{(1)}_{u^2} \frac{\partial}{\partial v^2} H^{(0)} + 2 W^{(0)}_{u^2} \frac{\partial}{\partial v^2} H^{(1)} \\
&+ H^{(1)} \frac{\partial}{\partial u^2} W^{(0)}_{v^2} + H^{(1)} \frac{\partial}{\partial v^2} W^{(0)}_{u^2} + W^{(1)}_{u^2} W^{(0)}_{v^2} \frac{\partial}{\partial v^2} W^{(0)}_{u^2} \\
&+ 2 W^{(0)}_{v^2} \frac{\partial}{\partial u^2} W^{(0)}_{v^2} + \frac{\partial^2}{\partial u^1 \partial u^2} W^{(0)}_{v^2} - W^{(0)}_{v^2} \frac{\partial}{\partial u^1} W^{(1)}_{u^2} \\
&+ \frac{\partial^2}{\partial u^1 \partial v^2} W^{(0)}_{u^2} - \frac{1}{2} \left(W^{(1)}_{u^2} W^{(0)}_{v^2} \right)^2 - W^{(1)}_{u^2} W^{(0)}_{v^2} \frac{\partial}{\partial u^2} W^{(0)}_{v^2} \\
&+ \frac{\partial}{\partial v^2} W^{(0)}_{u^2} \frac{\partial}{\partial u^2} W^{(0)}_{v^2} - \frac{1}{2} \left(\frac{\partial}{\partial v^2} W^{(0)}_{u^2} \right)^2 - \frac{1}{2} \left(\frac{\partial}{\partial u^2} W^{(0)}_{v^2} \right)^2
\end{aligned}
\tag{14}
$$

$$
0 = \frac{\partial}{\partial v^2} H^{(1)} - \frac{1}{2} W^{(1)}_{u^2} \frac{\partial}{\partial v^2} W^{(0)}_{v^2} + \frac{1}{2} \frac{\partial^2}{\partial (v^2)^2} W^{(0)}_{u^2} - \frac{1}{2} \frac{\partial^2}{\partial v^2 \partial u^2} W^{(0)}_{v^2} \tag{15}
$$

$$
\begin{aligned}
0 = -\frac{1}{2} \frac{\partial}{\partial u^1} W^{(1)}_{u^2} + \frac{\partial}{\partial u^2} H^{(1)} - \frac{1}{2} W^{(0)}_{v^2} \left(W^{(1)}_{u^2} \right)^2 + \frac{1}{2} W^{(1)}_{u^2} \frac{\partial}{\partial v^2} W^{(0)}_{u^2} \\
+ \frac{1}{2} W^{(0)}_{v^2} \frac{\partial}{\partial u^2} W^{(1)}_{v^2} - \frac{1}{2} \frac{\partial^2}{\partial v^2 \partial u^2} W^{(0)}_{u^2} + \frac{1}{2} \frac{\partial^2}{\partial (u^2)^2} W^{(0)}_{v^2}
\end{aligned}
\tag{16}
$$

$$
0 = \frac{\partial}{\partial u^2} W^{(1)}_{u^2} - \frac{1}{2} \left(W^{(1)}_{u^2} \right)^2 \tag{17}
$$

The form of the metric (6) is preserved under the transformations (10), (11). Applying the transformation (10), without loss of generality we can set $W^{(0)}_{v^2} = 0$.

One solution of eqn. (17) is $W^{(1)}_{u^2} = 0$; the resulting solutions are then [9]:

$$
\begin{aligned}
H^{(1)}(u^1, u^2, v^2) &= \alpha(u^1, u^2) + \beta(u^1, v^2), \\
W^{(0)}_{u^2}(u^1, u^2, v^2) &= 2\alpha(u^1, u^2) v^2 - 2 \int \beta(u^1, v^2) dv^2 + \gamma(u^1, u^2), \\
H^{(0)}(u^1, u^2, v^2) &= 2\beta(u^1, v^2) v^2 \int \alpha du^2 - 2\beta(u^1, v^2) u^2 \int \beta(u^1, v^2) dv^2 + \beta(u, y) \int \gamma(u, u^2) du^2 \\
&\quad + v^2 \int \frac{\partial}{\partial u^1} \alpha(u^1, u^2) du^2 - u^2 \int \frac{\partial}{\partial u^1} \beta(u^1, v^2) dv^2 + F_1(u^1, u^2) + F_2(u^1, v^2)
\end{aligned}
$$

In particular, we obtain the special neutral 4D, Ricci-flat, VSI solution of [1] in which all of the W's are zero, $H^1 = H^2 = 0$, and $H^{(0)}(u^1, u^2, v^2) = F_1(u^1, v^2) + F_2(u^1, u^2)$ in terms of the arbitrary functions F_1 and F_2.

Assuming that $W^{(0)}_{v^2} \neq 0$, eqn. (17) can then be solved as follows:

$$W^{(1)}_{u^2} = -2\frac{1}{\alpha(u^1) + u^2} \equiv -2f$$

Next we integrate eqn. (15) with respect to v^2, which gives

$$W^{(1)}_{u^2} H^{(1)} + \beta(u^1, u^2) = 2\frac{\partial}{\partial u^2} H^{(1)},$$

where β is an arbitrary function. We then obtain

$$\frac{\partial}{\partial u^2} H^{(1)} + f H^{(1)} = \beta.$$

This has the solution

$$H^{(1)} = e^{-I}\left(\int \beta e^I du^2 + \gamma(u^1, v^2)\right),$$

where $I = \int f du^2 = \ln(f^{-1})$ and γ is an arbitrary function (and any integrating factor that is a function of (u^1, v^2) can be absorbed into γ). For notational simplicity we redefine the arbitrary function β as $\beta(u^1, u^2) \equiv f \int \beta/f du^2$. We can now write down the solution for $H^{(1)}$ in the general form:

$$H^{(1)}(u^1, u^2, v^2) = \beta(u^1, u^2) + f\gamma(u^1, v^2).$$

We can now use eqn. (14) to obtain:

$$W^{(0)}_{u^2} = \delta(u^1, u^2)v^2 - 2f(u^1, u^2)\int \gamma(u^1, v^2)\mathrm{d}v^2 + \eta(u^1, u^2)$$

Equation (16) now contains only the terms $W^{(1)}_{u^2}, H^{(1)}$, and $W^{(0)}_{u^2}$, and hence simply puts constraints on the arbitrary (integration) functions. After some manipulation we obtain

$$\beta(u^1, u^2) = \frac{1}{2}\delta(u^1, u^2) - \frac{\mathrm{d}\alpha(u^1)}{\mathrm{d}u^1} f(u^1, u^2) + \int f(u^1, u^2)\delta(u^1, u^2)\mathrm{d}u^2 + \epsilon(u^1)$$

Using the remaining coordinate freedom (11) we can now choose $\epsilon(u^1) = 0$, and hence we obtain the solutions

$$H^{(1)} = \frac{1}{2}\delta - \frac{\mathrm{d}\alpha(u^1)}{\mathrm{d}u^1} f + \int f\delta \mathrm{d}u^2 + f\gamma$$

$$W^{(0)}_{u^2} = \delta y - 2f\int \gamma \mathrm{d}v^2 + \eta$$

$$W^{(1)}_{u^2} = -2f$$

Substituting these expressions into (14) we obtain the solution

$$
\begin{aligned}
H^{(0)} &= f\Big(\int \gamma \mathrm{d}v^2 \int \delta \mathrm{d}u^2 - \left(2\int (\gamma)^2 \mathrm{d}v^2\right)\left(\int f \mathrm{d}u^2\right) \\
&+ \left(\int v^2 \frac{\partial \gamma}{\partial v^2} \mathrm{d}v^2\right)\left(\int \delta \mathrm{d}u^2\right) \\
&- 2\left(\int \left(\frac{\partial \gamma}{\partial v^2}\right)\left(\int \gamma \mathrm{d}v^2\right)\mathrm{d}v^2\right)\int f \mathrm{d}u^2 \\
&+ \left(\int \frac{\partial \gamma}{\partial v^2}\mathrm{d}v^2\right)\int \eta \mathrm{d}u^2 \\
&+ \left(\frac{\mathrm{d}\alpha(u^1)}{\mathrm{d}u^1}\right)\left(\int \gamma \mathrm{d}v^2\right)\int f \mathrm{d}u^2 \\
&- \left(\int \gamma \mathrm{d}v^2\right)\left(\int\int f\delta \mathrm{d}u^2 \mathrm{d}u^2\right) \\
&- \left(\int \gamma \mathrm{d}v^2\right)\int \frac{1}{f}\left(\frac{\partial f}{\partial u^1}\right)\mathrm{d}u^2 \\
&- u^2\left(\int \frac{\partial \gamma}{\partial u^1}\mathrm{d}v^2\right) - \frac{1}{2}v^2\left(\frac{\mathrm{d}\alpha(u^1)}{\mathrm{d}u^1}\right)\left(\int \delta \mathrm{d}u^2\right) \\
&+ \frac{1}{2}v^2\left(\int \frac{1}{f}\left(\int f\delta \mathrm{d}u^2\right)\delta \mathrm{d}u^2\right) \\
&+ \frac{1}{2}v^2\left(\int \frac{1}{f}\left(\frac{\partial \delta}{\partial u^1}\right)\mathrm{d}u^2\right) + F_1(u^1, v^2)\Big) \\
&+ F_2(u^1, u^2)
\end{aligned}
\tag{18}
$$

4.2 Spacelike/timelike case

Setting $R_{ab} = 0$ and using the coordinate freedom (12) to set $W_T^{(0)} = 0$, we obtain the following equations:

$$
2X^2 \frac{\partial}{\partial T} H^{(1)} + \frac{\partial}{\partial X}\left(X^2 \frac{\partial}{\partial T} W_X^{(0)}\right) = 0 \tag{19}
$$

$$
2X^2 \frac{\partial^2}{\partial T^2} H^{(1)} = \frac{\partial}{\partial X}\left(2X^2 \frac{\partial}{\partial X} H^{(1)} - 2W_X^{(0)}\right) \tag{20}
$$

$$
X^2 \frac{\partial^2}{\partial T^2} W_X^{(0)} = -\left(2X^2 \frac{\partial}{\partial X} H^{(1)} - 2W_X^{(0)}\right) \tag{21}
$$

$$
\begin{aligned}
&2X^2 \frac{\partial^2}{\partial X^2} H^{(0)} - 4X \frac{\partial}{\partial X} H^{(0)} + 4H^{(0)} - 2X^2 \frac{\partial^2}{\partial T^2} H^{(0)} = \\
&\quad W_X^{(0)}\left(2X^2 \frac{\partial}{\partial X} H^{(1)} - 2W_X^{(0)}\right) + 2X^2 W_X^{(0)} \frac{\partial}{\partial X} H^{(1)} + 2X^2 H^{(1)} \frac{\partial}{\partial X} W_X^{(0)} \\
&\qquad - X^2\left(\frac{\partial}{\partial T} W_X^{(0)}\right)^2 + 2X^2 \frac{\partial^2}{\partial u \partial X} W_X^{(0)}
\end{aligned}
\tag{22}
$$

Equations (19)-(21) do not contain $H^{(0)}$. We can solve those equations first for $H^{(1)}$ and $W_X^{(0)}$, and then use the fourth equation to obtain $H^{(0)}$. Note also that the first three equations are not independent. Taking the derivative of (19) with respect to T and using eqn. (21) gives eqn. (20).

Integrating eqn. (19) with respect to T gives

$$H^{(1)} = \alpha(u,X) - \frac{1}{2X^2}\frac{\partial}{\partial X}\left(X^2 W_X^{(0)}\right), \tag{23}$$

where $\alpha(u,X)$ is an arbitrary function. Replacing $H^{(1)}$ in eqn. (21) with this form gives an equation for $W_X^{(0)}$ in terms of an arbitrary function. After applying the general identity $x^2\frac{\partial^2}{\partial x^2}f + 2x\frac{\partial}{\partial x}f = x\frac{\partial^2}{\partial x^2}(xf)$ for any function $f(x,\ldots)$, we obtain the more compact equation for $W_X^{(0)}$:

$$\left(\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial T^2}\right)\left(XW_X^{(0)}\right) = 2X\frac{\partial}{\partial X}\alpha(u,X) \tag{24}$$

A change of variables $A = X + T$, $B = X - T$ is then used. The operator on the left turns into a single second order mixed derivative, so that

$$4\frac{\partial^2}{\partial A\partial B}\left(XW_X^{(0)}\right) = 2X\frac{\partial}{\partial X}\alpha(u,X), \tag{25}$$

and thus after integration

$$W_X^{(0)} = \frac{1}{X}\left(\iint \frac{X}{2}\frac{\partial}{\partial X}\alpha(u,X)dAdB + \beta(u,A) + \gamma(u,B)\right), \tag{26}$$

where β and γ are arbitrary functions. Since the integrand in this expression is not a function of T, integration with respect to A and B can be replaced by integration with respect to X using $dA = dB = 2dX$:

$$\begin{aligned}\iint \frac{X}{2}\frac{\partial}{\partial X}\alpha(u, X = \frac{A+B}{2})dAdB &= 2\iint X\frac{\partial \alpha}{\partial X}dXdX \\ &= 2X\int \alpha dX - 4\iint \alpha dXdX \end{aligned}\tag{27}$$

where integration by parts has been used. It is convenient to redefine our arbitrary function as $\alpha(u,X) \equiv 2\iint \alpha dXdX$ and rescale β and γ, where we keep the same name for notational simplicity. Hence,

$$W_X^{(0)} = \frac{2}{X}\left(\beta(u,X+T) + \gamma(u,X-T) - \alpha(u,X)\right) + \frac{\partial \alpha}{\partial X}(u,X). \tag{28}$$

It is straightforward to write down the solution for $H^{(1)}$ by using eqn. (23). Keeping in mind the redefinition of α, we obtain:

$$\begin{aligned}H^{(1)} = \frac{1}{X^2}\Big(&\alpha(u,X) - \beta(u,X+T) - \gamma(u,X-T) \\ &- X\frac{\partial \beta}{\partial X}(u,X+T) - X\frac{\partial \gamma}{\partial X}(u,X-T)\Big).\end{aligned} \tag{29}$$

We now have solved eqns. (19) and (21). Eqn. (20) is automatically satisfied since it is not independent of the other two, as was discussed earlier. All that remains is to solve eqn. (22).

The terms involving derivatives of $H^{(0)}$ with respect to X can be simplified by using differential identities, by using the previously solved equations and by using

$$W_X^{(0)}\left(2X^2\frac{\partial}{\partial X}H^{(1)} - 2W_X^{(0)}\right) - X^2\left(\frac{\partial}{\partial T}W_X^{(0)}\right)^2$$

$$= -\frac{1}{2}X^2\frac{\partial^2}{\partial T^2}\left(W_X^{(0)}\right)^2 \tag{30}$$

where we have used eqn. (21) and the chain rule. Furthermore,

$$2X^2W_X^{(0)}\frac{\partial}{\partial X}H^{(1)} + 2X^2H^{(1)}\frac{\partial}{\partial X}W_X^{(0)} = 2X^2\frac{\partial}{\partial X}\left(W_X^{(0)}H^{(1)}\right) \tag{31}$$

Therefore, we can write down the more concise equation for $H^{(0)}$:

$$\left(\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial T^2}\right)\left(\frac{H^{(0)}}{X}\right) =$$
$$\frac{1}{X}\left(\frac{\partial^2}{\partial u\partial X}W_X^{(0)} + \frac{\partial}{\partial X}\left(W_X^{(0)}H^{(1)}\right) - \frac{1}{4}\frac{\partial^2}{\partial T^2}\left(W_X^{(0)}\right)^2\right) \tag{32}$$

For given functions $W_X^{(0)}$ and $H^{(1)}$ this equation can be integrated to solve for $H^{(0)}$.

5 Some exact CSI solutions

We can also consider 4D neutral signature spaces which are CSI, for which all of the curvature invariants are constant [5]. By solving the appropriate components of the Riemann tensor equal to constants (see eqns. (15) - (22) in [1]), and using similar techniques to those in [5], we can find examples of 4D neutral CSI metrics. In general the Einstein equations that need to be solved are quite complicated. Therefore, let us consider two simple examples for illustration. In particular, the neutral signature "Siklos metrics" (in which the only non-zero independent invariant is the Ricci scalar, which is constant) give rise to relatively simple equations that can be solved completely.

Lorentzian CSI spacetimes are known to be solutions of supergravity theory when supported by appropriate bosonic fields [6], and it is likely that neutral signature CSI spaces are also of physical interest [7, 8].

5.1 CSI example 1:

Let us write the CSI metric as:

$$\mathrm{d}s^2 = 2\boldsymbol{\ell}^1 \otimes \boldsymbol{n}^1 - \exp(2KX)\mathrm{d}T \otimes \mathrm{d}T + \mathrm{d}X \otimes \mathrm{d}X,$$

where

$$\begin{aligned} \boldsymbol{\ell}^1 &= \mathrm{d}u & (33)\\ \boldsymbol{n}^1 &= \mathrm{d}v + [v^2\sigma + vH^{(1)}(u,T,X) + H^{(0)}(u,T,X)]\mathrm{d}u & (34)\\ &+[Av + W_X^{(0)}(u,T,X)]\mathrm{d}X + [Bv + W_T^{(0)}(u,T,X)]\exp(KX)\mathrm{d}T \end{aligned}$$

There are many cases which lead to an Einstein space. In particular, we consider the simple case:

$$A = -2K, \quad B = 0, \quad \sigma = 0.$$

Writing down the Einstein conditions, we obtain $\Lambda = -3K^2$ from the diagonal terms of the Ricci tensor, and differential equations from the off-diagonal terms. All of these have simple curvature structure; in particular, $C_{abcd}C^{abcd} = 0$.

Explicitly, the metric has the form

$$\begin{aligned} \mathrm{d}s^2 &= 2\mathrm{d}u \otimes (\mathrm{d}v + [v^2 + vH^{(1)} + H^{(0)}]\mathrm{d}u \\ &+ [-2Kv + W_X^{(0)}]\mathrm{d}X \\ &+ W_T^{(0)}\exp(KX)\mathrm{d}T) - \exp(2KX)\mathrm{d}T \otimes \mathrm{d}T + \mathrm{d}X \otimes \mathrm{d}X \qquad (35) \end{aligned}$$

where $H^{(1)}, H^{(0)}, W_X^{(0)}, W_T^{(0)}$ are functions of u, T, X. The conditions for the metric to be an Einstein space yield the following system of partial differential equations:

$$\begin{aligned} & 4\frac{\partial H^{(1)}}{\partial X}W_X^{(0)}\exp(2KX) + 2K\exp(2KX)\frac{\partial H^{(0)}}{\partial X} \\ +\; & 4K^2\exp(2KX)H^{(0)} + 2\frac{\partial W_T^{(0)}}{\partial X} \\ +\; & 2\frac{\partial W_T^{(0)}}{\partial X}\exp(KX)\frac{\partial W_X^{(0)}}{\partial T} \\ +\; & 2K\exp(2KX)\frac{\partial W_X^{(0)}}{\partial u} - (W_T^{(0)})^2K^2\exp(2KX) \\ -\; & 2v\frac{\partial^2 H^{(1)}}{\partial X^2}\exp(2KX) + 2\frac{\partial W_T^{(0)}}{\partial X}\exp(2KX)W_T^{(0)}K \\ -\; & 6K\exp(2KX)v\frac{\partial H^{(1)}}{\partial X} + 2K\exp(2KX)H^{(1)}W_X^{(0)} \\ -\; & 2W_T^{(0)}K\exp(KX)\frac{\partial W_X^{(0)}}{\partial T} \\ +\; & 2H^{(1)}\frac{\partial W_X^{(0)}}{\partial X}\exp(2KX) - 4\frac{\partial H^{(1)}}{\partial T}W_T^{(0)}\exp(KX) \\ -\; & 2H^{(1)}\frac{\partial W_T^{(0)}}{\partial T}\exp(KX) \\ -\; & 2\frac{\partial^2 W_T^{(0)}}{\partial u\partial T}\exp(KX) + 2v\frac{\partial^2 H^{(1)}}{\partial T^2} \\ +\; & 2\frac{\partial^2 H^{(0)}}{\partial T^2} - \left(\frac{\partial W_T^{(0)}}{\partial X}\right)^2 - \left(\frac{\partial W_X^{(0)}}{\partial T}\right)^2 \\ -\; & \frac{\partial^2 H^{(0)}}{\partial X^2}\exp(2KX) + 2\frac{\partial^2 W_X^{(0)}}{\partial u\partial X}\exp(2KX) = 0 \qquad (36) \end{aligned}$$

$$
\begin{aligned}
&2\frac{\partial H^{(1)}}{\partial T} - K\frac{\partial W_T^{(0)}}{\partial X}\exp(KX) + 2W_T^{(0)}K^2\exp(KX) + \\
&\qquad K\frac{\partial W_X^{(0)}}{\partial T} - \frac{\partial^2 W_T^{(0)}}{\partial X^2}\exp(KX) + \frac{\partial^2 W_X^{(0)}}{\partial X \partial T} = 0
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
& 2\frac{\partial H^{(1)}}{\partial X}\exp(2KX) - \frac{\partial^2 W_T^{(0)}}{\partial X\partial T}\exp(KX) \\
+ \quad & \frac{\partial W_T^{(0)}}{\partial T}K\exp(KX) + \frac{\partial^2 W_X^{(0)}}{\partial T^2} = 0
\end{aligned} \tag{38}
$$

The v dependence in eqn. (36) gives

$$
\begin{aligned}
- \quad & 2\frac{\partial^2 H^{(1)}}{\partial X^2}\exp(2KX) - 6K\exp(2KX)\frac{\partial H^{(1)}}{\partial X} \\
+ \quad & 2\frac{\partial^2 H^{(1)}}{\partial T^2} = 0
\end{aligned} \tag{39}
$$

The form of the metric (35) is preserved under the transformations (12), (13). Thus, without loss of generality, we can set $W_T^{(0)} = 0$. We note that $H^{(0)}$ does not appear in equations (37), (38), (39). We first use eqns. (37) and (38) to solve for $W_X^{(0)}$ and $H^{(1)}$, and then use eqn. (39) to put constraints on these solutions.

First we can integrate eqn. (37) with respect to T to get

$$
2H^{(1)} + kW_X^{(0)} + \frac{\partial W_X^{(0)}}{\partial X} + \alpha = 0, \tag{40}
$$

where $\alpha = \alpha(u, X)$ is an arbitrary function. Now, using using eqns. (40) and (38) we can solve for $H^{(1)}$ and $W_X^{(0)}$:

$$
\begin{aligned}
H^{(1)} &= \frac{1}{2}\bar{\alpha} + \frac{1}{2}\frac{\partial \bar{\alpha}}{\partial X} - \frac{1}{2}\alpha - \frac{1}{2}KC_1T \\
&- \frac{1}{2}KC_2 - \frac{1}{2}K\gamma - \frac{1}{4}KC_3T^2 + \frac{1}{4}C_3K^{-1}\exp(-2KX)
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
W_X^{(0)} &= \frac{1}{2}C_3T^2 + C_1T + \frac{1}{2}C_3K^{-2}\exp(2KX) \\
&- \bar{\alpha} + \rho K^{-1}\exp(-KX) + \gamma
\end{aligned} \tag{42}
$$

where

$$
\bar{\alpha} = \int \frac{\int \exp(KX)\frac{\partial \alpha}{\partial X}dX}{\exp(KX)}dX \tag{43}
$$

and $\rho = \rho(u), \alpha = \alpha(u, X), \gamma = \gamma(u)$ are arbitrary functions. We can now put constraints on the form of (41) using eqn. (39). After simpification, eqn. (39) becomes

$$2C_3K(-\exp(-2KX) + 1) = 0 \tag{44}$$

from which we conclude that $C_3 = 0$ (since $K = 0$ is not consistent with eqn. (36)).

Finally, we employ the remaining transformational freedom to set $\gamma(u) = 0$. Thus we are left with the following differential equation for $H^{(0)}$:

$$\begin{aligned}
& 2C_2K^2\exp(2KX)\bar{\alpha} + 2C_2K\exp(2KX)\frac{\partial\bar{\alpha}}{\partial X} - 2K\exp(2KX)\frac{\partial\bar{\alpha}}{\partial X}\bar{\alpha} \\
- & C_2K\exp(2KX)\alpha + K\exp(2KX)\alpha\bar{\alpha} - C_1^2K^2\exp(2KX)T^2 \\
- & C_1^2 + 2C_1K^2\exp(2KX)\bar{\alpha}T + 2C_1K\exp(2KX)\frac{\partial\bar{\alpha}}{\partial X}T - C_1K\exp(2KX)\alpha T \\
- & 2C_1C_2K^2\exp(2KX)T - \exp(2KX)\left(\frac{\partial\bar{\alpha}}{\partial X}\right)^2 \\
- & 2\exp(KX)\int\exp(KX)\frac{\partial^2\alpha}{\partial u\partial X}\alpha\mathrm{d}X - 2K\exp(2KX)\bar{\alpha} - K^2\exp(2KX)\bar{\alpha}^2 \\
- & C_2^2K^2\exp(2KX) + \exp(2KX)\frac{\partial\bar{\alpha}}{\partial X}\alpha + 2\frac{\partial^2H^{(0)}}{\partial T^2} - 2\exp(2KX)\frac{\partial^2H^{(0)}}{\partial X^2} \\
+ & 2K\exp(2KX)\frac{\partial H^{(0)}}{\partial X} + 4K^2\exp(2KX)H^{(0)} = 0
\end{aligned} \tag{45}$$

A particularly simple subcase is the case where $H^{(1)} = W_X^{(0)} = W_T^{(0)} = 0$, whence we obtain the special solution $H^{(0)} = C\exp(-KX)$, which is the Kaigorodov case.

5.2 CSI example 2:

The CSI metric can be written

$$\mathrm{d}s^2 = 2(\boldsymbol{\ell}^1 \otimes \boldsymbol{n}^1 + \boldsymbol{\ell}^2 \otimes \boldsymbol{n}^2)$$

where

$$\begin{aligned}
\boldsymbol{\ell}^1 &= \mathrm{d}u & (46)\\
\boldsymbol{n}^1 &= \mathrm{d}v + [Av^2 + vH^{(1)}(u,U,V) + H^{(0)}(u,U,V)]\mathrm{d}u & (47)\\
&\quad +[vV\beta + W_U^{(0)}(u,U,V)]\mathrm{d}U + [\alpha v/V + W_V^{(0)}(u,U,V)](\mathrm{d}V + BV^2\mathrm{d}U) & \\
\boldsymbol{\ell}^2 &= \mathrm{d}U & (48)\\
\boldsymbol{n}^2 &= \mathrm{d}V + BV^2\mathrm{d}U, & (49)
\end{aligned}$$

and A, B, α and β are constants.

We look for Einstein spaces in the special case:

$$A = 0, \quad \alpha = -2, \quad \beta = 2B$$

(where B is not specified). The metric then has the explicit form

$$\begin{aligned}
\mathrm{d}s^2 &= 2\mathrm{d}u \otimes (\mathrm{d}v + [vH^{(1)} + H^{(0)}]\mathrm{d}u + [2BvV + W_U^{(0)}]\mathrm{d}U \\
&+ [-2\frac{v}{V} + W_V^{(0)}](dV + BV^2\mathrm{d}U)) + 2\mathrm{d}U \otimes \mathrm{d}V + 2BV^2\mathrm{d}U \otimes \mathrm{d}U
\end{aligned} \tag{50}$$

The conditions for the metric to be Einstein yields the following system of (independent) differential equations:

$$\begin{aligned}
& -4BV^4\frac{\partial^2 H^{(0)}}{\partial V^2} + 4v\frac{\partial^2 H^{(1)}}{\partial V \partial U}V^2 + 2BV^4\frac{\partial^2 W_V^{(0)}}{\partial u \partial V} \\
- & \quad 4BV^4 v\frac{\partial^2 H^{(1)}}{\partial V^2} - 16BV^3 v\frac{\partial H^{(1)}}{\partial V} + 2H^{(1)}V^4 B\frac{\partial W_V^{(0)}}{\partial V} \\
+ & \quad 4BV^3 H^{(1)}W_V^{(0)} + 2\frac{\partial W_U^{(0)}}{\partial V}V^4 B\frac{\partial W_V^{(0)}}{\partial V} - 4BV^3\frac{\partial W_V^{(0)}}{\partial V}W_U^{(0)} \\
- & \quad 2BV^4\frac{\partial W_V^{(0)}}{\partial V}\frac{\partial W_V^{(0)}}{\partial U} + 4BV^4\frac{\partial H^{(1)}}{\partial V}W_V^{(0)} + 4Vv\frac{\partial H^{(1)}}{\partial U} \\
+ & \quad 8BV^2 H^{(0)} - 2H^{(1)}V^2\frac{\partial W_U^{(0)}}{\partial V} - 2H^{(1)}V^2\frac{\partial W_V^{(0)}}{\partial U} + 4BV^3\frac{\partial W_V^{(0)}}{\partial u} \\
- & \quad 4\frac{\partial H^{(1)}}{\partial V}V^2 W_U^{(0)} - 4\frac{\partial H^{(1)}}{\partial U}W_V^{(0)}V^2 - 4\frac{\partial W_U^{(0)}}{\partial V}VW_U^{(0)} \\
- & \quad 2\frac{\partial W_U^{(0)}}{\partial V}V^2\frac{\partial W_V^{(0)}}{\partial U} + B^2V^6\left(\frac{\partial W_V^{(0)}}{\partial V}\right)^2 + 4W_U^{(0)}\frac{\partial W_V^{(0)}}{\partial U}V \\
+ & \quad \left(\frac{\partial W_U^{(0)}}{\partial V}\right)^2 V^2 - 4V\frac{\partial H^{(0)}}{\partial U} + \left(\frac{\partial W_V^{(0)}}{\partial U}\right)^2 + 4(W_U^{(0)})^2 \\
- & \quad 2\frac{\partial^2 W_V^{(0)}}{\partial u \partial U}V^2 - 2\frac{\partial^2 W_U^{(0)}}{\partial u \partial V}V^2 + 4\frac{\partial^2 H^{(0)}}{\partial V \partial U}V^2 = 0 \qquad (51)
\end{aligned}$$

$$\begin{aligned}
& 2\frac{\partial H^{(1)}}{\partial U}V - 2\frac{\partial H^{(1)}}{\partial V}BV^3 + 4B^2V^4\frac{\partial W_V^{(0)}}{\partial V} - 2BVW_U^{(0)} \\
- & \quad \frac{\partial W_V^{(0)}}{\partial U}BV^2 - \frac{\partial^2 W_U^{(0)}}{\partial V \partial U}V - 2BV^3\frac{\partial^2 W_V^{(0)}}{\partial V \partial U} + 2\frac{\partial W_U^{(0)}}{\partial U} \\
+ & \quad \frac{\partial^2 W_V^{(0)}}{\partial U^2}V + BV^3\frac{\partial^2 W_U^{(0)}}{\partial V^2} + B^2V^5\frac{\partial W_V^{(0)}}{\partial V^2} = 0 \qquad (52)
\end{aligned}$$

$$\begin{aligned}
& 4BV^3\frac{\partial W_V^{(0)}}{\partial V} - 2W_U^{(0)} - 2\frac{\partial W_V^{(0)}}{\partial U}V + 2\frac{\partial H^{(1)}}{\partial V}V^2 \\
+ & \quad \frac{\partial^2 W_U^{(0)}}{\partial V^2}V^2 + BV^4\frac{\partial^2 W_V^{(0)}}{\partial V^2} - \frac{\partial^2 W_V^{(0)}}{\partial V \partial U}V^2 = 0 \qquad (53)
\end{aligned}$$

The v-dependency in eqn. (51) gives

$$4\frac{\partial^2 H^{(1)}}{\partial V \partial U}V^2 - 4BV^4\frac{\partial^2 H^{(1)}}{\partial V^2} - 16BV^3\frac{\partial H^{(1)}}{\partial V} + 4V\frac{\partial H^{(1)}}{\partial U} = 0 \qquad (54)$$

where $H^{(1)} = H^{(1)}(u, U, V), H^{(0)} = H^{(0)}(u, U, V), W_U^{(0)} = W_U^{(0)}(u, U, V), W_V^{(0)} = W_V^{(0)}(u, U, V)$.

Equations (52), (53), (54) do not contain any $H^{(0)}$ terms. The form of the metric (50) is invariant under the transformations (10) and (11). Explicitly, transformation (10) becomes [10]:

$$(u', v', U', V') = (u, v + h(u, U, V), U, V) \tag{55}$$

where

$$H'^{(1)} = H'^{(1)}, \quad H'^{(0)} = H'^{(0)} - hH^{(1)} - h_{,u} \tag{56}$$

We then define

$$\overline{W}_V^{(0)} = -\frac{2h}{V} + W_V^{(0)} - h_{,V}, \tag{57}$$

which is just rewriting the form of an arbitrary function (no v's are introduced). Then the other metric components transform as:

$$\left(-2\frac{v}{V} + W_V^{(0)}\right)' = -2\frac{v}{V} + \overline{W}_V^{(0)} \tag{58}$$

$$\left(W_U^{(0)} + BV^2 W_V^{(0)}\right)' = W_U^{(0)} - h_{,U} + BV^2 h_{,V} + BV^2\overline{W}_V^{(0)} \tag{59}$$

and so the form of the metric is preserved under transformation (10) (with all arbitrary functions remaining arbitrary).

Therefore, we use this freedom to set $W_V^{(0)} = 0$ and obtain the solutions

$$H^{(1)} = \alpha(u) + \frac{\beta(u)}{V^3} \tag{60}$$

$$W_U^{(0)} = V^2\gamma(u, U) + \frac{(-4B\beta(u)U + \delta(u))}{V} + \frac{3}{2}\frac{\beta(u)}{V^2} \tag{61}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary functions. Substituting the above solutions into (51) gives a differential equation for $H^{(0)}$:

$$\begin{aligned}
& -4BV^2\frac{\partial^2 H^{(0)}}{\partial V^2} + 8BV^2 H^{(0)} - 2\left(\alpha + \frac{\beta}{V^3}\right)V^2\left(-\frac{(-4B\beta U + \delta)}{V^2} + 2V\gamma - \frac{3\beta}{V^3}\right) \\
+ \quad & \frac{12\beta\left(\frac{-4B\beta U + \delta}{V} + V^2\gamma + \frac{3\beta}{2V^2}\right)}{V^2} + 4\frac{\partial^2 H^{(0)}}{\partial V \partial U}V^2 \\
- \quad & 4\left(-\frac{-4B\beta U + \delta}{V^2} + 2V\gamma - \frac{3\beta}{V^3}\right)V\left(\frac{-4B\beta U + \delta}{V} + V^2\gamma + \frac{3\beta}{2V^2}\right) \\
+ \quad & V^2\left(-\frac{-4B\beta U + \delta}{V^2} + 2V\gamma - \frac{3\beta}{V^3}\right)^2 \\
- \quad & 2V^2\left(-\frac{-4B\frac{d\beta}{dU}U + \frac{d\delta}{du}}{V^2} + 2V\frac{\partial\gamma}{\partial u} - \frac{3\frac{d\beta}{du}}{V^3}\right) = 0
\end{aligned} \tag{62}$$

6 Conclusion

We have presented a number of new exact 4D neutral signature VSI and CSI solutions that are of interest in the twistor approach to string theory [7] and particularly spaces admitting parallel spinors [8].

We note that the neutral signature case is different from the Lorentzian and Riemannian cases. The Riemannian VSI and CSI cases are locally flat and locally homogeneous, respectively, while the Lorentzian VSI and CSI cases lead to the possibility of non-homogeneous Kundt spacetimes. The neutral signature case also leads to non-homogeneous Kundt spaces, but also allows for an additional "null" degree of freedom. Thus, comparing the Riemannian, Lorentzian and the neutral signature VSI and CSI cases, the neutral signature case allows for the richest variety of spaces.

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